THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2023-2024 Term 1
Suggested Solutions of Homework Assignment 3

Q1

(a). The image of X is $\{4, 2, 1, 0, -1, -2\}$.

$$P(X = 4) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$

$$P(X = 2) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$

$$P(X = 1) = \frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}$$

$$P(X = 0) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$P(X = -1) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

$$P(X = -2) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{4}{13}$$

(b).

$$E(X) = \sum_{x:p(x)>0} x \cdot p(x) = 4 \cdot \frac{6}{91} + 2 \cdot \frac{8}{91} + 1 \cdot \frac{32}{91} + 0 \cdot \frac{1}{91} - 1 \cdot \frac{16}{91} - 2 \cdot \frac{4}{13} = 0$$

The expected value of the money we are going to get for playing 100 games is $(0-2) \times 100 = -200$ dollars.

(c). It it not fair. The game is biased against us.

Q2

The possible values of X are $1, 2, 3, 4, 5, 6$ and the probabilities that X takes on each of these values are

$$\begin{aligned}
P(X=1) &= \frac{5 \cdot 9!}{10!} = \frac{1}{2} \\
P(X=2) &= \frac{5 \cdot 5 \cdot 8!}{10!} = \frac{5}{18} \\
P(X=3) &= \frac{5 \cdot 4 \cdot 5 \cdot 7!}{10!} = \frac{5}{36} \\
P(X=4) &= \frac{5 \cdot 4 \cdot 3 \cdot 5 \cdot 6!}{10!} = \frac{5}{84} \\
P(X=5) &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 5!}{10!} = \frac{5}{252} \\
P(X=6) &= \frac{5!5!}{10!} = \frac{1}{252} \\
P(X=7) &= P(X=8) = P(X=9) = P(X=10) = 0
\end{aligned}$$

Q3

First, find the mass probability functions of X and Y. Let $\Omega = \{40, 33, 25, 50\}$.

$$
\begin{aligned}
P(X = 40) &= \frac{40}{148} \\
P(X = 33) &= \frac{33}{148} \\
P(X = 25) &= \frac{25}{148} \\
P(X = 50) &= \frac{50}{148} \\
P(Y = i) &= \frac{1}{4},\ i \in \Omega
\end{aligned}
$$

Then we are going to calculate the expectation and variance of X and Y.

$$E(X) = \sum_{k\in\Omega} kP(X = k) = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} \approx 39.28$$

$$E(Y) = \sum_{k\in\Omega} kP(Y = k) = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 37$$

$$E\left(X^2\right) = \sum_{k\in\Omega} k^2P(X = k) = 40^2 \cdot \frac{40}{148} + 33^2 \cdot \frac{33}{148} + 25^2 \cdot \frac{25}{148} + 50^2 \cdot \frac{50}{148}$$

$$\operatorname{Var}(X) = E\left(X^2\right) - E(X)^2 \approx 82.20$$

$$E\left(Y^2\right) = \sum_{k\in\Omega} k^2P(Y = k) = 40^2 \cdot \frac{1}{4} + 33^2 \cdot \frac{1}{4} + 25^2 \cdot \frac{1}{4} + 50^2 \cdot \frac{1}{4}$$

$$\operatorname{Var}(Y) = E\left(Y^2\right) - E(Y)^2 = 84.5$$

Q4

(a). By simple observation, we can see the event $\{X = 1\}$ is equal to $\{(H, T), (T, H)\}$.

$$P(X = 1) = P\{(H, T), (T, H)\} = 0.6 \times (1 - 0.7) + (1 - 0.6) \times 0.7 = 0.46$$

(b). Similarly, $\{X = 2\} = \{(H, H)\}$. Then

$$\begin{aligned}
E(X) &= \sum_{k=0}^{2} kP(X = k) \\
&= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) \\
&= 1 \cdot 0.46 + 2 \cdot (0.6 \times 0.7) \\
&= 1.3
\end{aligned}$$

Q5

(a). Denote the event "first coin is flipped" by C_1, with C_2 defined similarly. Let X be the number of heads out of 10 tosses.

$$\begin{aligned}
P(X = 7) &= P(X = 7 \mid C_1) P(C_1) + P(X = 7 \mid C_2) P(C_2) \\
&= \left[\binom{10}{7} \cdot 4^7 \cdot 6^3\right] \frac{1}{2} + \left[\binom{10}{7} .7^7 \cdot 3^3\right] \frac{1}{2} \\
&= [.0425].5 + [.2668].5 \\
&= .1547
\end{aligned}$$

(b). By conditioning on the outcome of the first flip, we update the probability (now evenly split between coins 1 and 2) that coin 1 is being flipped. Let H_1 denote the event "the first flip is heads".

$$\begin{aligned}
P(C_1 \mid H_1) &= \frac{P(H_1 \mid C_1) P(C_1)}{P(H_1 \mid C_1) P(C_1) + P(H_1 \mid C_2) P(C_2)} \\
&= \frac{.4 \times .5}{.4 \times .5 + .7 \times .5} \\
&= \frac{4}{11}
\end{aligned}$$

Our updated probabilities are now: $P(C_1 \mid H_1) = \frac{4}{11}$ and $P(C_2 \mid H_1) = \frac{7}{11}$. Then

$$
\begin{aligned}
&P(X = 7 \mid H_1) \\
&= \frac{P(\{X = 7\} \cap H_1)}{P(H_1)} \\
&= \frac{P(\{X = 7\} \cap C_1 \cap H_1) + P(\{X = 7\} \cap C_2 \cap H_1)}{P(H_1)} \\
&= \frac{P(\{X = 7\} \cap C_1 \cap H_1)}{P(C_1 \cap H_1)} \cdot \frac{P(C_1 \cap H_1)}{P(H_1)} + \frac{P(\{X = 7\} \cap C_2 \cap H_1)}{P(C_2 \cap H_1)} \cdot \frac{P(C_2 \cap H_1)}{P(H_1)} \\
&= P(X = 7 \mid C_1 \cap H_1) \cdot P(C_1 \mid H_1) + P(X = 7 \mid C_2 \cap H_1) \cdot P(C_2 \mid H_1) \\
&= \left[\binom{9}{6} \cdot 4^6 \cdot 6^3 \right] \frac{4}{11} + \left[\binom{9}{6} .7^6 \cdot 3^3 \right] \frac{7}{11} \\
&= [.0743].3636 + [.2668].6364 \\
&= .1968
\end{aligned}
$$

Q6

The number of potential interviewees who consent to the interview X is a binomial random variable, with $n = 5$ and $p = \frac{2}{3}$. Q: What is the probability that each of the 5 people consents to the interview?

(a). $P(X=5) = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^0 = \left(\frac{2}{3}\right)^5 = \frac{32}{243} = .1317$

(b). Now $X \sim \text{Binom}\left(8, \frac{2}{3}\right)$

$$\begin{aligned} P(X \geq 5) &= \sum_{k=5}^{8} \begin{pmatrix} 8 \\ k \end{pmatrix} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{8-k} \\ &= .7414 \end{aligned}$$

(c). We are asked for the probability that the 6th potential interviewee will be the 5th to consent.

$$\begin{aligned} P(X=6) &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} \left(\frac{2}{3}\right)^5 \frac{1}{3} \\ &= \frac{160}{729} = .2195 \end{aligned}$$

(d).

$$\begin{aligned} P(X=7) &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^2 \\ &= \frac{160}{729} = .2195 \end{aligned}$$

Q7

As the expected value of a function of a discrete random variable X equals $\sum_i f(x_i) P\{X = x_i\}$, c^X has as an expected value of

$$\begin{aligned} E\left[c^X\right] &= c^1 P\{X=1\} + c^{-1} P\{X=-1\} \\ E\left[c^X\right] &= c(p) + \left(\frac{1}{c}\right)(1-p) \\ E\left[c^X\right] &= cp + \frac{1-p}{c} \end{aligned}$$

Setting this equal to 1,

$$\begin{aligned}
cp + \frac{1-p}{c} &= 1 \\
pc^2 + 1 - p &= c \\
pc^2 - c + 1 - p &= 0
\end{aligned}$$

If $p = 0$, then $c = 1$ (disregarded). If $p \neq 0$, we solve this quadratic equation and get

$$\begin{aligned}
c &= \frac{1 \pm \sqrt{(-1)^2 - 4p(1-p)}}{2p} \\
c &= \frac{1 \pm \sqrt{(2p-1)^2}}{2p} \\
c &= \frac{1 \pm (2p-1)}{2p}
\end{aligned}$$

Solving the equation above, we get $c = \frac{1}{p} - 1$ or $c = 1$ (disregarded).

Q8

$$\begin{aligned}
E\left(\frac{1}{X+1}\right) &= \sum_{k=0}^{n} \frac{1}{k+1} P(X=k) \\
&= \sum_{k=0}^{n} \frac{1}{k+1} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1} (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \left(\sum_{i=0}^{n+1} \frac{(n+1)!}{i!(n+1-i)!} p^i (1-p)^{n+1-i} - (1-p)^{n+1} \right) \\
&= \frac{1}{(n+1)p} \left((p + (1-p))^{n+1} - (1-p)^{n+1} \right) \\
&= \frac{1 - (1-p)^{n+1}}{(n+1)p}
\end{aligned}$$

Q9

Note that a Poisson r.v. has the parameter $\lambda > 0$ and $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$ Fix a $k \in \{0, 1, 2, \ldots\}$ and let $f_k(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$. If $k = 0$, note that $f_0(\lambda) = e^{-\lambda}, \lambda > 0$, so no maximum is attained for $k = 0$. If $k > 0$, then

$$
\begin{aligned}
f_k'(\lambda) =& \frac{e^{-\lambda}}{k!}\left(k\lambda^{k-1} - \lambda^k\right) \\
& \begin{cases} > 0 & \text{if } \lambda < k \\ = 0 & \text{if } \lambda = k \\ < 0 & \text{if } \lambda < k \end{cases}
\end{aligned}
$$

Hence $\lambda = k$ maximizes $P(X = k)$.

Q10

First, according to the expectation of a function of random variables, we have

$$
\begin{aligned}
E\left(X^n\right) &= \sum_{k=1}^{\infty} k^n \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lambda \sum_{k=1}^{\infty} k^{n-1} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} \\
&= \lambda \sum_{i=0}^{\infty} (i+1)^{n-1} \frac{\lambda^i e^{-\lambda}}{i!} \\
&= \lambda E\left((X+1)^{n-1}\right).
\end{aligned}
$$

We have known $E(X) = \lambda$. Then by the formula proved above,

$$
\begin{aligned}
E\left(X^2\right) &= \lambda E(X+1) = \lambda(E(X)+1) = \lambda^2 + \lambda \\
E\left(X^3\right) &= \lambda E\left((X+1)^2\right) \\
&= \lambda\left(E\left(X^2 + 2X + 1\right)\right) \\
&= \lambda\left(E\left(X^2\right) + 2E(X) + 1\right) \\
&= \lambda\left(\lambda^2 + \lambda + 2\lambda + 1\right) \\
&= \lambda^3 + 3\lambda^2 + \lambda.
\end{aligned}
$$